Chapter 3

- parts of a circle.

3.1 properties of circles.
- area of a sector of a circle.

the area of the smaller sector can be found by the following formula:

\[ A = \frac{\theta}{360} \pi r^2, \text{ given } \theta \text{ in degrees, or} \ A = \frac{\theta}{2} \pi r^2, \text{ given } \theta \text{ in radians} \]

- the Chord Right Bisector Property (RBP).

\[ OM \parallel AB \iff AM = MB \]

ie. the centre of a circle always lies on the perpendicular bisector of any chord.

proof:
I. given \( OM \parallel AB \)

because \( \angle OAM = \angle OBM \) (HS =),
therefore, \( AM = MB \) (\( = \) s)
II. given \( AM = MB \)

because \( \triangle OAM \cong \triangle OBM \) (SSS),
\[
\square = \square \quad (\equiv \square s)
\]
therefore, \( \square = \square = 90^\circ \)

ex. prove that \( XY \) is the right bisector of \( PQ \).

\[
\triangle XYP \cong \triangle XQY \quad (SSS)
\]
therefore \( \triangle PXM = \triangle QXM \)
so \( \triangle XPM = \triangle XMQ \) (SAS \( \equiv \))
ergo, \( \triangle XMP = \triangle XMQ = 90^\circ \) and \( PM = MQ \) (\( \equiv \square s \))
QED

- the Equal Chords Property (ECP).

\[
AB = CD \text{ iff } d_1 = d_2
\]

proof:
I. given \( AB = CD \)

\[
ND = MB
\]
\( \triangle OBM \cong \triangle ODN \)
\( OM = ON \) (\( \equiv \square s \))

II. given \( OM = ON \)

\( \triangle OMB \cong \triangle ONC \) (HS \( \equiv \))
therefore, \( MB = NC \)
similarly, \( AM = DN \)
thus \( AM + MB = DN + NC \)
\[
AB = DC
\]
3.2 angles in a circle.

- some definitions.

- Angles in a Circle Properties (ACP).

proof:

because  \( OC = OA = OB \), \( \angle OAC \) and \( \angle OCB \) are isosceles
thus, \( \angle ACO = \angle CAO \) and \( \angle OBC = \angle OCB \)
by \( \text{EAT} \), \( \angle DOA = 2(\angle OAC) \) and \( \angle DOB = 2(\angle OCB) \)
\[ \angle AOB = 2(\angle OAC) + 2(\angle OCB) \]
\[ \angle AOB = 2(\angle ACB) \quad \text{QED.} \]
ex. find the value of $w$, $x$, $y$, $z$.

$$w = 40^\circ, x = 70^\circ, y = 60^\circ, z = 30^\circ$$

ex. prove $\angle ACE$ is isosceles.

in $\angle EBD$, $ED = BD$

therefore $\angle E = \angle T (\text{TTT})$

but $\angle D = \angle C$, ergo $\angle D = \angle C$

$\angle D = \angle C$ because $\angle D$ and $\angle C$ are subtended by the same arc

therefore $\angle D = \angle C$ by transitivity, hence $\angle ACE$ is isosceles. QED.

ex. prove that $AB = BC$.

because $OA$ and $OC$ are radii of a common circle, $OA = OC$.

$OB$ is common to both $\angle OBC$ and $\angle OBA$.

because $OA$ is a diameter, by ACP, $\angle OBA = 90^\circ$.

because $OB \parallel AC$, $\angle AB = BC$ (RBP).

ex. determine the size of the central angle subtended by an arc of $2\theta$ if the circle’s radius is 6.

$$\frac{\theta}{360^\circ} = \frac{2\theta}{12\theta}$$

$$\theta = 12\theta$$

therefore $\frac{\theta}{360^\circ} = \frac{1}{6}$

$\theta = 60^\circ$

3.3 cyclic quadrilaterals.

- concyclic points are points which lie on the circumference of the same circle.

- cyclic polygons have concyclic vertices.

ex. prove that all triangles are cyclic given any $\angle ABC$

ie. prove $OA = OB = OC$

clearly, $OA = OB$ (RBT)

similarly, $OB = OC$ (RBT)

by transitivity, therefore $OA = OB = OC$ QED.
ex. prove that if a quadrilateral $ABCD$ is cyclic, then its opposite angles are supplementary.

\[2\alpha + 2\beta = 360^\circ\]
\[\alpha + \beta = 180^\circ\]

QED.

- Cyclic Quadrilateral Properties.

1. opposite angles are supplementary
2. any exterior angle is equal to the opposite interior angle
3. a side subtends equal angles at the remaining vertices
   ie. $\angle BAC = \angle BDC$ or $\angle CAD = \angle CBD$

ex. in a cyclic quadrilateral $ABCD$, $AB = AD$, $\angle BCD = 110^\circ$, $\angle BAC = 30^\circ$. find $\angle ABC$.

$\alpha = 40^\circ \ (CQP)$ and $\beta = 55^\circ$

but $CD$ subtends both $\angle CAD$ and $\angle CBD$

so $\angle CAD = \angle CBD = 40^\circ$

therefore, $\angle ABC = 55^\circ + 40^\circ + 95^\circ$

ex. show that $BCED$ is cyclic.

in $\angle ABE$, $\angle A = 70^\circ$

in $\angle ABC$,

$50^\circ + 70^\circ + 2\gamma = 180^\circ$

$\gamma = 30^\circ$

$\angle BEC = 100^\circ = \angle BDC$

because side $BC$ subtends equal angles at $D$ and $E$, therefore $BCED$ is cyclic.
3.5 tangent properties.

- Tangent/Radius Properties (TRP)

\[ \text{AB is a tangent iff } OP \perp AB \]

proof:
I. given \( AB \) is a tangent

assume \( \angle OPA = 90^\circ \)

WLOG let \( \angle OPA < 90^\circ \)

construct \( Q \) on \( AB \) such that \( \angle OQP = \angle OPA \)

ergo, \( OQ = OP \) (ITT) \( \equiv \)

therefore \( OP \perp AB \)

II. given \( OP \perp BA \)

assume \( AB \) is a secant.

clearly, \( OQ = OP \)

so \( \angle OQP = \angle OPQ = 90^\circ \)

\( \equiv \)

therefore, \( AB \) is a tangent.

QED.

- Tangent from a Point to a circle Property (TPP)

\[ \text{IF } PA \text{ and } PB \text{ are tangent segments,} \]
\[ \text{THEN } PA = PB \]

proof:

in \( \triangle AOP \), \( \triangle BOP \)

\( AO = BO \) (radii of a common circle)

\( OP \) is common and \( \angle OAP = \angle OBP = 90^\circ \) (TRP)

therefore, \( \triangle AOP \equiv \triangle BOP \) (HS)

whence \( AP = BP \), QED.
ex. if $PA = 15$, determine the perimeter of $\triangle PCD$.

Clearly, $DE = DB$ and $EC = CA$ (TPP)

The perimeter of $\triangle PCD$

$= PD + DE + PC$
$= PD + DE + EC + PC$
$= (PD + DB) + (CA + PC)$
$= PB + PA$
$= 30$

- Tangent Chord Property (TCP)

The angle formed by a chord and a tangent is equal to the inscribed angle subtended by the chord in the opposite segment.

I.e. $\angle BAD = \angle BXA$
$\angle BAC = \angle BYA$

Proof:

Clearly, $\angle = \angle (ACP)$, and $\angle YBA = 90^\circ$ (ACP)

$\angle + \angle = 90^\circ$ (SATT)

But $\angle + \angle = 90^\circ$ (TRP)

Whence $\angle = \angle = \angle$. QED.

- Intersecting Chords Property (ICP)

If two chords $AB$ and $CD$ intersect at $E$

Then $AE \parallel EB = CE \parallel ED$

Proof:

$\angle AEC = \angle DEB$ (opposite angles) and $\angle ACD = \angle ABD$ (subtended by common arc)

Therefore $\angle ACE \sim \angle DBE$ (AA~).

Thus $\frac{AE}{ED} = \frac{CE}{EB}$ ($\sim \parallel$s)

Whence $AE \parallel EB = CE \parallel ED$ (POP) QED.
- Intersecting Secants Property (ISP)

IF two secants \( AB \) and \( CD \) meet at \( P \)
THEN \( AP \cap PB = CP \cap PD \)

- Corollary to ISP

\[ AP \cap PB = (PT)^2 \]

ex. prove that \( ADQP \) is cyclic.

clearly, \( \angle XBC \sim \angle XDA \) and \( \angle = \angle \)

since \( \frac{XQ}{QB} = \frac{XP}{PC} = \frac{1}{1} \)

therefore \( QP \parallel BC \) (DST)

so \( \angle = \angle \) (PLT, corresponding angles)

thus \( \angle = \angle \)

whence \( ADQP \) is cyclic because \( \angle \) and \( \angle \) are subtended by side \( AP \)

for review questions, see:

p. 112 #1-7
p. 108 #1, 2, 4, 5, 7, 9-12, 14-16, 18, 21